KS; David E. Manes SUNY College at Oneonta, Oneonta, NY; Ken Korbin, NewYork, NY; Kee-Wai Lau, Hong Kong, China; Toshihiro Shimizu, Kawasaki, Japan; Albert Stadler, Herrliberg, Switzerland, and the proposers.

- **5358:** *Proposed by Arkady Alt, San Jose, CA*

Prove the identity $\displaystyle\sum_{k=1}^{m} k\binom{m+1}{k+1} r^{k+1} = (r+1)^m(mr-1)+1.$

Solution 1 by Ángel Plaza, University of Las Palmas de Gran Canaria, Spain

$$\begin{aligned}
(r+1)^m(mr-1)+1 &= \sum_{k=0}^{m} m\binom{m}{k} r^{k+1} - \sum_{k=1}^{m}\binom{m}{k} r^k \\
&= \sum_{k=1}^{m} m\binom{m}{k} r^{k+1} - \sum_{k=1}^{m-1}\binom{m}{k+1} r^{k+1} \\
&= mr^{m+1} + \sum_{k=1}^{m-1}\left(m\binom{m}{k} - \binom{m}{k+1}\right) r^{k+1} \\
&= mr^{m+1} + \sum_{k=1}^{m-1} k\binom{m+1}{k+1} r^{k+1} \\
&= \sum_{k=1}^{m} k\binom{m+1}{k+1} r^{k+1}
\end{aligned}$$

where we have used that $m\binom{m}{k} - \binom{m}{k+1} = k\binom{m+1}{k+1}$.

Solution 2 by Anastasios Kotronis, Athens, Greece

We have

$$(1+r)^m = \sum_{k=0}^{m}\binom{m}{k} r^k \tag{1}$$

and differentiating

$$mr(1+r)^{m-1} = \sum_{k=0}^{m} k\binom{m}{k} r^k. \tag{2}$$

Now

$$\begin{aligned}
\sum_{k=1}^{m} k\binom{m+1}{k+1} r^{k+1} &= \sum_{k=2}^{m+1}(k-1)\binom{m+1}{k} r^k = \sum_{k=2}^{m+1} k\binom{m+1}{k} r^k - \sum_{k=2}^{m+1}\binom{m+1}{k} r^k \\
&\overset{(2),(1)}{=\!=} (m+1)r(1+r)^m - (m+1)r - (1+r)^{m+1} + 1 + (m+1)r \\
&= (r+1)^m(mr-1)+1.
\end{aligned}$$

Solution 3 by Paolo Perfetti, Department of Mathematics, University Tor Vergata, Rome, Italy

Proof Induction. Let $m=1$. We have

$$\binom{2}{2}r^2 = (r+1)(r-1)+1$$

which clearly holds.

Let's suppose it is true for $2 \le m \le n-1$. For $m=n$ we have

$$\sum_{k=1}^{m+1} k\binom{m+2}{k+1}r^{k+1} = (m+1)r^{m+2} + \sum_{k=1}^{m} k\left[\binom{m+1}{k+1}+\binom{m+1}{k}\right]r^{k+1} =$$
$$= (m+1)r^{m+2} + (r+1)^m(mr-1)+1\sum_{k=1}^{m} k\binom{m+1}{k}r^{k+1} \qquad (1)$$

$$\binom{m+2}{k+1} = \binom{m+1}{k+1}+\binom{m+1}{k}$$

and the induction hypothesis have been used. Moreover

$$\sum_{k=1}^{m} k\binom{m+1}{k}r^{k+1} \underbrace{=}_{k+1=p} r\sum_{p=0}^{m-1}(p+1)\binom{m+1}{p+1}r^{p+1} =$$
$$= r\sum_{p=1}^{m-1} p\binom{m+1}{p+1}r^{p+1} + r\sum_{p=0}^{m-1}\binom{m+1}{p+1}r^{p+1} =$$
$$= r\sum_{p=1}^{m} p\binom{m+1}{p+1}r^{p+1} - mr^{m+2} \underbrace{+}_{p+1=q} r\sum_{q=0}^{m+1}\binom{m+1}{q}r^{q} - r - r^{m+2}$$

The induction hypotheses and the Newton–binomial yield that it is equal to

$$r\left((r+1)^m(mr-1)+1\right) - mr^{m+2} + r(1+r)^{m+1} - r - r^{m+2}.$$

By inserting in (1) we get

$$(m+1)r^{m+2} + \left((r+1)^m(mr-1)+1\right)(r+1) - (m+1)r^{m+2} + r(1+r)^{m+1} - r =$$
$$= (r+1)^{m+1}(mr-1) + (r+1) + r(1+r)^{m+1} - r =$$
$$= (r+1)^{m+1}((m+1)r-1) - r(1+r)^{m+1} + (r+1) + r(1+r)^{m+1} - r =$$
$$= (r+1)^{m+1}((m+1)r-1)+1.$$

and the proof is complete.

Solution 4 by David Stone and John Hawkins, Georgia Southern University, Statesboro, GA

Here we differentiate the given sum to get the Binomial Theorem, then integrate to get the desired sum.

Let $f(r) = \sum_{k=1}^{m} \binom{m+1}{k+1} r^{k+1} = \sum_{k=1}^{m} k \frac{(m+1)!}{(k+1)k(k-1)!(m+1-k-1)!} r^{k+1}$,

so,

$$\begin{aligned} f'(r) &= \sum_{k=1}^{m} k \frac{(k+1)(m+1)!}{(k+1)k(k-1)!(m-k)!} r^k \\ &= \sum_{k=1}^{m} k \frac{(m+1)!}{(k-1)!(m-k)!} r^k \\ &= \sum_{k=0}^{m-1} mk \frac{(m+1)!}{(k-1)!(m-1-k)!} r^k, \text{by reindexing} \\ &= m(m+1) \sum_{k=0}^{m-1} \frac{(m-1)!}{k!(m-1-k)!} r^k, \\ &= m(m+1)r \sum_{k=0}^{m-1} \binom{m-1}{k} r^k \\ &= m(m+1)r(r+1)^{m-1} \text{ by the Binomial Theorem.} \end{aligned}$$

Now we can integrate by parts to fine $f(r)$:

$$\begin{aligned} f(r) &= \int m(m+1)(r(r+1)^{m-1} dr \\ &= m(m+1) \int r(r+1)^{m-1} dr \\ &= m(m+1) \left[\frac{1}{m} r(r+1)^m - \int \frac{1}{m}(r+1) dr \right] \\ &= m(m+1) \left[\frac{1}{m} r(r+1)^m - \frac{1}{m} \frac{(r+1)^{m+1}}{m+1} \right] + C \\ &= m(m+1) \left\{ \frac{(r+1)^m}{m} \frac{(mr-1)}{m+1} \right\} + C \\ &= (r+1)^m (mr-1) + C \end{aligned}$$

Using the initial condition $f(0) = 0$ we find $C = 1$, so $f(r) = (r+1)^m(mr-1)+1$, as desired.

Editor's note: David and John also submitted a second solution to this problem that was similar to Solution 2 above.

Also solved by Dionne Bailey, Elsie Campbell and Charles Diminnie, Angelo State University San Angelo TX; Charles Burnette (Graduate student, Drexel University), Philadelphia, PA; Jerry Chu (student, Saint George's School), Spokane, WA; Bruno Salgueiro Fanego, Viveiro, Spain; Ed Gray, Highland Beach, FL; G. C. Greubel, Newport News, VA; Paul M. Harms, North Newton, KS; Kee-Wai Lau, Hong Kong, China; Moti Levy, Rehovot, Israel; Carl Libis, Eastern Connecticut State University, Willimantic, CT David E. Manes SUNY College at Oneonta, Oneonta, NY; Toshihiro Shimizu, Kawasaki, Japan; Albert Stadler, Herrliberg, Switzerland, and the proposers.

5359: *Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain.*

Let a, b, c be positive real numbers. Prove that

$$\sqrt[4]{15a^3b+1} + \sqrt[4]{15b^3c+1} + \sqrt[4]{15c^3a+1} \le \frac{63}{32}(a+b+c) + \frac{1}{32}\left(\frac{1}{a^3}+\frac{1}{b^3}+\frac{1}{c^3}\right).$$

Solution 1 by Arkady Alt, San Jose, CA

Since $15a^3b+1$ can be represented as $(2a)^3 \cdot \dfrac{15b+\dfrac{1}{a^3}}{8}$ then by AM-GM Inequality we obtain

$$\sum_{cyc}\sqrt[4]{15a^3b+1} = \sum_{cyc}\sqrt[4]{(2a)^3\cdot\frac{15b+\dfrac{1}{a^3}}{8}} \le \sum_{cyc}\frac{3\cdot(2a)+\dfrac{15b+\dfrac{1}{a^3}}{8}}{4}$$

$$= \sum_{cyc}\frac{48a+15b+\dfrac{1}{a^3}}{32}$$

$$\le \frac{63}{32}(a+b+c) + \frac{1}{32}\left(\frac{1}{a^3}+\frac{1}{b^3}+\frac{1}{c^3}\right).$$

Solution 2 by Albert Stadler, Herrliberg, Switzerland

We first claim that

$$\sqrt[4]{11+15x^4} \le \frac{63}{32}x + \frac{1}{32x^3}, \ x>0. \qquad (1)$$